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Sensitivity analysis of permeability parameters for flows on Barcelona networks

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ABSTRACT

We consider the problem of optimizing vehicular traffic flows on an urban network of Barcelona type, i.e. square network with streets of not equal length. In particular, we describe the effects of variation of permeability parameters, that indicate the amount of flow allowed to enter a junction from incoming roads.
On each road, a model suggested by Helbing et al. (2007) [11] is considered: free and congested regimes are distinguished, characterized by an arrival flow and a departure flow, the latter depending on a permeability parameter. Moreover we provide a rigorous derivation of the model from fluid dynamic ones, using recent results of Bretti et al. (2006) [3]. For solving the dynamics at nodes of the network, a Riemann solver maximizing the through flux is used, see Coclite et al. (2005) [4] and Helbing et al. (2007) [11].
The network dynamics gives rise to complicate equations, where the evolution of fluxes at a single node may involve time-delayed terms from all other nodes. Thus we propose an alternative hybrid approach, introducing additional logic variables. Finally we compute the effects of variations on permeability parameters over the hybrid dynamics and test the obtained results via simulations.

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1. Introduction

Traffic flows in urban areas are classical example of material flows. Most of times, they are not diffusive or going on in continuous space, but organized in networks. A suitable mathematical model for flows can be given by conservation laws and, as we deal with partial differential equations with nonlinear characteristics, it turns out that this is not a trivial task. On one side, there are quite serious mathematical issues and, on the other, one could ask if they could represent some typical phenomena of daily life (see for example, the modelling of traffic flows along road sections [5,8,16,20] and the problem of finding solutions for road networks, see [4,7]). In general, for the treatment of flows on networks several approaches were considered, see [2,6,7,10,12–15,17].

To overcome the difficulties encountered by complex fluid dynamic models, in this work we make use of a simplified two-phase model for flows on roads, proposed by last author and collaborators (descriptions of such model and similar ones can be found in [9–11]). The road network is decomposed into road sections (links) of homogeneous capacity and nodes for their connections. Traffic dynamics along the links is assumed to follow the Lighthill–Whitham–Richards model (briefly LWR, see [18,21]), but with a simplified representation reducing the PDE approach to a delayed-ode ones. For every road, two regimes are considered: Free and congested; the lengths of the corresponding areas determine the exact dynamics of cars. Here we prove a rigorous derivation of the model from recent results for LWR on networks (see [3]).

The two-phase model, beside providing a simplified description, is also useful in order to describe some effects observed in real traffic, among which the transitions from free to congested traffic flows due to lack of capacity, the propagation speeds of vehicles in congested traffic, and spill-over effects. The adoption of free and congested areas for each road also allows the study of dynamics connected to traffic jams, which are treated in two different ways: Either by computing the number of cars that are delayed or by determining fronts and ends of traffic jams.

Flows at nodes are regulated via permeability parameters γ, which prescribe the amount of cars allowed to enter the junction from incoming roads. In [11], these parameters are assumed to be always zero or one (to model a traffic light, with permeability equal to one corresponding to green light). Here we consider permeability as a function $\gamma(t) \in [0, 1]$. In [11] some solutions are proposed for connecting, merging, diverging or intersection points of the homogeneous road sections. The latter are based on two rules introduced in [4,7]:

(A) the incoming traffic distributes to outgoing roads according to fixed (statistical) distribution coefficients;
(B) drivers behave in order to maximize the through flux.

Here we consider permeability parameters as controls in order to optimize the dynamics of complex networks of Barcelona type, characterized by square networks and roads of different lengths. More precisely, we focus on the minimization of a suitable cost functional, that consists in the sum of queue lengths (i.e. number of delayed vehicles or lengths of congested areas). We show that the formation of queues at a road junction highly influences the urban traffic evolution of the whole network. In particular, under some assumptions, the network description can generate a "nested" equation, impossible to be represented in a compact analytical way. For this reason we introduce additional logic variables. As a result, the obtained system is of hybrid nature. The continuous variables are the queue lengths, with arrival and departure flows being time-varying variables, which affect in a delayed way the continuous variables. Logic variables, corresponding to emptying of queues or filling of the whole road segments, are influenced by the continuous variables and influence the flows. Finally, the network structure implies relations among flows, given explicitly by the solution at nodes.

Inspired by the celebrated Pontryagin Maximum Principle [1], we consider needle variations of control parameters, which in turn permit sensitivity analysis with respect to permeability parameters. The hybrid structure of the dynamics gives rise to rich phenomena in terms of variations of the involved variables. In particular, needle variations in the control produce consequent needle variations in the flows, which can be computed looking at node dynamics. Moreover, the latter provoke jumps in variations of continuous variables. Finally, switching variations show up for logic variables.

After providing various theoretical results to describe this rich phenomenon, we implement a Runge–Kutta numerical algorithm for the hybrid dynamics for verification. A simple needle variation of a permeability parameter provokes a wealth of variations in the other quantities at nearby nodes. These results indicate that the hybrid approach is the correct one: on one side it permits the description of the network evolution with nodes dynamics having separate equations and, on the other side, it keeps all characteristics of the original system.

The paper is organized as follows. The used model is illustrated in Section 2, together with a rigorous derivation from the LWR model on networks. Section 3 contains notations for Barcelona networks and description of the dynamics at nodes. In Section 4, we introduce the optimal control problem and show the nested equations raising up. To avoid this drawback, logic variables are defined leading to a hybrid dynamics. Section 5 provides theoretical results for sensitivity analysis of permeability parameters, while Section 6 contains relative numerical results. The paper ends with Section 7 discussing achieved results and indicating future perspectives.

2. Model for flows on roads

We consider a multiscale model for road networks, which was originally introduced in [9,10]. The model encompasses macroscopic quantities such as inflows and outflows as well as microscopic ones as the number of delayed vehicles. First we list the basic assumptions under which this approach can be used and then provide two different descriptions of jams (or queues) evolutions. Finally, we show a derivation of the used model from the classical Lighthill–Whitham–Richards (briefly LWR) fluid dynamic ones [18,19], using recent extensions of the model to networks [4]. This section is focused on the dynamics for each single road forming the network, while the dynamics at nodes is analyzed for the special case of Barcelona type networks in next section.

The following assumptions are made:

- A1 A first order approach (such as LWR) gives a sufficiently good description of the dynamics.
- A2 On each road section, the fundamental diagram (density–flux graph) can be well approximated by a triangular shape, with an increasing slope V_i^0 (i.e. maximum speed of vehicles) at low densities and a decreasing slope c in the congested regime. V_i^0 corresponds to the free speed or speed limit on road section i, while

$$-c = -\frac{1}{\rho_{\max} T}$$

  is the dissolution speed. Here $\rho_{\max}$ denotes the maximum vehicle density in vehicle queues, while T is the safe time headway which is constant along the road section.
- A3 Who enters a road section first exits first (FIFO principle). That is, overtaking is assumed to be negligible.
- A4 Each road section is characterized by a first subsection in free phase (low density) and a second subsection in congested phase (high density).

Assumptions A1–A3 are strong assumptions, which however are usually satisfied by homogeneous road sections in urban areas. The last assumption A4 is consequence of the previous ones as we will show in Section 2.2.

The *arrival flow* $A_j(t)$ denotes the inflow of vehicles into the upstream end of road section j, while $O_j(t)$ is the *departure flow*, i.e. the flow of vehicles leaving road section j at its downstream end. The quantity

$$\widehat{Q}_j = \left(T + \frac{1}{V_j^0 \rho_{\max}}\right)^{-1} = \frac{\rho_{\max}}{1/c + 1/V_j^0} \tag{1}$$

represents the maximum in- or outflow of road section j.

All the above quantities refer to flows *per lane*. I_j is the number of lanes and L_j the length of road section j. Moreover, the length $l_j(t) \leqslant L_j$ represents the length of the congested area on link j (measured from the downstream end), and ΔN_j is the number of stopped or delayed vehicles. Constraints can be given for the *actual* arrival and departure flows, given by the *potential arrival flows* $\widehat{A}_j(t)$ and the *potential departure flows* $\widehat{O}_i(t)$, respectively.

The actual arrival flow $A_j(t)$ is limited by the maximum inflow $\widehat{Q}_j$, if road section j is not fully congested ($l_j(t) < L_j$); otherwise ($l_j = L_j$), it is limited by $O_j(t - L_j/c)$ a time period L_j/c before.

Hence,

$$0 \leqslant A_j(t) \leqslant \widehat{A}_j(t) := \begin{cases} \widehat{Q}_j & \text{if } l_j(t) < L_j, \\ O_j(t - L_j/c) & \text{if } l_j(t) = L_j. \end{cases} \tag{2}$$

Moreover, the potential departure flow $\widehat{O}_i(t)$ of road section i is given by its permeability $\gamma_i(t)$ times the maximum outflow $\widehat{Q}_i$ from this road section, if vehicles are queued up ($\Delta N_i > 0$) and waiting to leave; otherwise ($\Delta N_i = 0$), the outflow is limited by the permeability times the arrival flow A_i a time period L_i/V_i^0 before. This gives the additional relationship:

$$0 \leqslant O_i(t) \leqslant \widehat{O}_i(t) := \gamma_i(t) \begin{cases} A_i(t - L_i/V_i^0) & \text{if } \Delta N_i(t) = 0, \\ \widehat{Q}_i & \text{if } \Delta N_i(t) > 0. \end{cases} \tag{3}$$

The permeability $\gamma_i(t)$ for traffic flows at the downstream end of section i can vary in the range $[0, 1]$. In case of a traffic light, $\gamma_i(t) = 1$ corresponds to a green light for road section i, while $\gamma_i(t) = 0$ corresponds to a red or amber light.

2.1. Traffic jams

Two views for traffic jams have been proposed in [11].

2.1.1. Case 1: number of delayed vehicles

Here, we deal with the more simple method, according to which the number of cars that are delayed are compared with free traffic. This corresponds to a situation in which the vehicles would not queue up along the road section, but at its downstream end. The number of delayed vehicles for the road section i, ΔN_i, evolves according to the following equation:

$$\Delta \dot{N}_i = A_i\left(t - \frac{L_i}{V_i^0}\right) - O_i(t). \tag{4}$$

As such method is used for describing traffic jams, in (2) we will replace $l_j(t) < L_j$ by $\Delta N_j(t) < N_j^{\max} := L_j \rho_{\max}$ and $l_j(t) = L_j$ by $\Delta N_j(t) = N_j^{\max}$. Notice that (4) may violate the maximal bound. Thus if $\Delta N_i(\bar{t}) = N_i^{\max}$ and $A_i(t - \frac{L_i}{V_i^0}) - O_i(t) > 0$ for $t > \bar{t}$, then we set $\Delta N_i(t) = N_i^{\max}$, namely the number of delayed vehicles is forced to its maximal value.

2.1.2. Case 2: jam formation and resolution

The formation and resolution of traffic jams is described by the shock wave equations, where we deal with two characteristic speeds V_i^0 (the free speed) and c (the jam resolution speed). The upstream end of a traffic jam, located at a place $l_i(t) \geqslant 0$, is moving at the speed

$$\frac{dl_i(t)}{dt} = -\frac{A_i(t - [L_i - l_i(t)]/V_i^0) - O_i(t - l_i(t)/c)}{A_i(t - [L_i - l_i(t)]/V_i^0)/V_i^0 - [1 - O_i(t - l_i(t)/c)/T]\rho_{\max}}. \tag{5}$$

2.2. A derivation of the model from LWR on networks

The LWR model consists of the single conservation law:

$$\rho_t + f(\rho)_x = 0,$$

where $\rho \in [0, \rho_{\max}]$ is the car density, while $f(\rho) = \rho v$ and the average velocity v depends only on the density ρ. From assumption A2 of previous section the flux is given by

$$f(\rho) = \begin{cases} V_i^0 \rho & \text{if } \rho \leqslant \sigma, \\ -c(\rho - \rho_{\max}) & \text{if } \rho \geqslant \sigma, \end{cases}$$

where $\sigma = \rho_{\max}/(1 + V_0^i T \rho_{\max})$ is the point of maximum flux. The density is said in free phase if $\rho \leqslant \sigma$ and in congested phase otherwise. The LWR model was extended to networks in [4]. Then some properties of solutions were proved in [3]. We make use of the latter to derive the proposed model from the LWR ones on networks.

In [3] it was proved the following:

Lemma 1. *Assume that $\rho(t,x)$ is a solution on the whole network with $\rho(0,\cdot) \equiv 0$. Then on every road i, parameterized by the interval $[0, L_i]$, and for every time t there exists $l_i(t)$ such that the density is in free phase for $x \in [0, L_i - l_i(t)[$ and in congested phase for $x \in]L_i - l_i(t), L_i]$.*

Thus assumption A4 is in fact guaranteed if the network starts from empty. This is precisely the case for all real urban networks due to low load during night hours.

Now the characteristic velocity in free phase is precisely V_0^i, while that in congested phase is given by $-c$. From this we can derive the jam dynamics.

In the first approach we assume that ideally the number of delayed vehicles is placed at the downstream endpoint of the road, thus the evolution of ΔN_i is given by the difference of the inflow, with a time delay, and the outflow. The time delay of the inflow is precisely given by the ratio between the road length L_i and the characteristic velocity in free phase, thus formula (4) follows.

In the second approach, at the point $L_i - l_i(t)$, separating the free subsection and the congested ones, a shock may occur. Thus to compute the velocity of the wave positioned at this point, we need to use the Rankine–Hugoniot relation (see [7]):

$$\frac{dl_i(t)}{dt} = \frac{f_- - f_+}{\rho_- - \rho_+},$$

where f_- is the flux to the left and f_+ to the right of $L_i - l_i(t)$, the same holding for ρ_- and ρ_+. The flux f_- is given by the inflow, computed with a time delay to cover the distance from the left endpoint of the road section with the characteristic velocity in free phase, thus given by $(L_i - l_i(t))/V_0^i$. On the other side f_+ is the outflow, computed with a time delay to cover the distance from the right endpoint with the characteristic velocity in congested phase, thus given by $l_i(t)/c$. Now the densities $\rho_\pm$ can be obtained by the relative fluxes dividing by the characteristic velocities. Finally we obtain precisely the relation (5).

3. Barcelona networks

A Barcelona network is given by a square network, with alternate directions (vertical: up and down, horizontal: left-to-right and right-to-left) and links of different lengths. We assume to have $\mathcal{N}$ rows and $\mathcal{M}$ columns. In particular, the network is denoted by the couple $(\mathcal{R}, \mathcal{J})$, where $\mathcal{R}$ and $\mathcal{J}$ indicate, respectively, the set of roads and junctions. Moreover $\mathcal{R} = \mathcal{R}_H \cup \mathcal{R}_V$, where $\mathcal{R}_H$ and $\mathcal{R}_V$ represent, respectively, the set of horizontal and vertical roads. Each node is identified by a couple $(i, j) \in \mathcal{J}$, where $i \in \mathcal{N}$ and $j \in \mathcal{M}$, and has two incoming and two outgoing roads. There are four

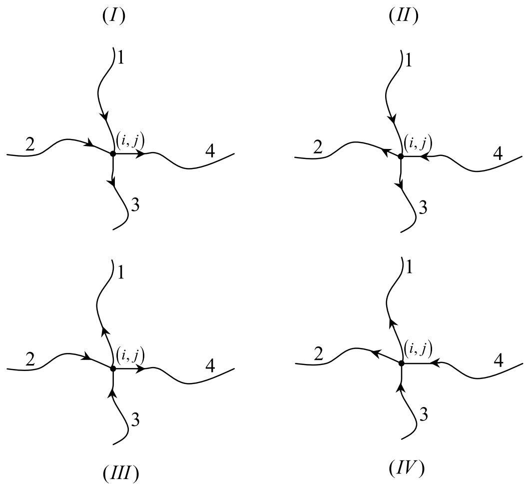

Fig. 1. Four cases of orientation for roads at node (i, j).

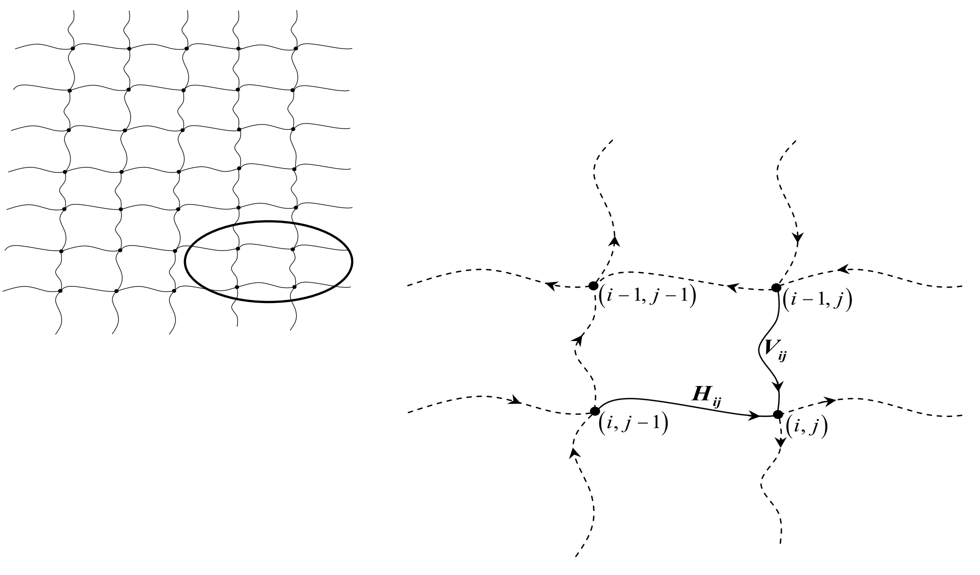

Fig. 2. Barcelona network (left) and zoom on a portion (right).

types of nodes: (I), (*II*), (*III*), and (*IV*), see Fig. 1 (1 and 3 indicate the vertical roads, while 2 and 4 the horizontal roads). In what follows, we will consider only nodes of type (I), the other cases being similar. At node (i, j) (of type (I)), vertical roads are labelled as V_{ij} (entering) and V_{i+1j} (exiting), while horizontal ones are labelled as H_{ij} (entering) and H_{ij+1} (exiting), as in Fig. 2. To simplify the notation, we make the following assumption:

(BH) All roads $V_{ij} \in \mathcal{R}_V$, $H_{ij} \in \mathcal{R}_H$, $i \in \mathcal{N}$, $j \in \mathcal{M}$, have the same maximum in- and outflow, i.e. $\widehat{Q}_k = \widehat{Q}$ $\forall k \in \mathcal{R}$ and free speed: $V_k^0 = V_0$ $\forall k \in \mathcal{R}$. Notice, however, that we consider possibly different lengths of roads $L_{V_{ij}}$ and $L_{H_{ij}}$.

Roads V_{ij} and H_{ij} have permeability parameters indicated by $\gamma_{V_{ij}}(t)$ and $\gamma_{H_{ij}}(t)$, respectively. Since the two roads share the same node (i,j), we assume:

$$0 \leqslant \gamma_{V_{ij}}(t) + \gamma_{H_{ij}}(t) \leqslant 1.$$

3.1. Dynamics at nodes

The dynamics at nodes is defined providing solutions to Riemann problems, i.e. Cauchy problems with initial data constant on each road. The map, associating to every initial data the corresponding fluxes emerging at the node, is called Riemann Solver and indicated by *RS*. The solution will depend on initial fluxes and on the number of delayed vehicles (resp. length of congested zone) of all roads meeting at the node.

Densities can also be reconstructed as in [4,7] using the fact that an incoming road k is in free phase only if $\Delta N_k = 0$ (resp. $l_k = 0$), while an outgoing one is in congested phase only if $\Delta N_k = N^{\max}$ (resp. $l = L$).

Let us provide two rules (given in [4,7]) to define a Riemann Solver:

(A) At each node $(i,j) \in \mathcal{J}$ drivers distribute according to fixed coefficients, given by a matrix

$$C = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ 1-\alpha_{ij} & 1-\beta_{ij} \end{pmatrix},$$

where $0 \leqslant \alpha_{ij}, \beta_{ij} \leqslant 1$ and α_{ij} (resp. β_{ij}) represents the percentage of traffic that, from road V_{ij} (resp. H_{ij}), goes to road H_{ij+1}.

(B) Respecting (A), drivers behave so as to maximize the flux through the node (i,j).

The outflows from incoming roads must satisfy the constraints:

$$0 \leqslant O_{V_{ij}} \leqslant \widehat{O}_{V_{ij}}, \qquad 0 \leqslant O_{H_{ij}} \leqslant \widehat{O}_{H_{ij}}, \tag{6}$$

moreover rule (A) imposes

$$0 \leqslant I_{H_{ij+1}} A_{H_{ij+1}} = \alpha_{ij} I_{V_{ij}} O_{V_{ij}} + \beta_{ij} I_{H_{ij}} O_{H_{ij}} \leqslant I_{H_{ij+1}} \widehat{A}_{H_{ij+1}}, \tag{7}$$

$$0 \leqslant I_{V_{i+1j}} A_{V_{i+1j}} = (1-\alpha_{ij}) I_{V_{ij}} O_{V_{ij}} + (1-\beta_{ij}) I_{H_{ij}} O_{H_{ij}} \leqslant I_{V_{i+1j}} \widehat{A}_{V_{i+1j}}. \tag{8}$$

The above constraints define a convex region of possible values in the $(O_{V_{ij}}, O_{H_{ij}})$ plane of fluxes from incoming roads. Then, thanks to rule (B) we determine uniquely the fluxes from incoming roads (assuming $\alpha_{ij} \neq \beta_{ij}$), while rule (A) determines uniquely the fluxes to outgoing roads $(A_{H_{ij+1}}, A_{V_{i+1j}})$. In the $(O_{V_{ij}}, O_{H_{ij}})$ plane, three situations may occur, labelled by *RS*1, *RS*2, and *RS*3 in Fig. 3. Let us make the simplifying assumptions $\Delta N_{V_{ij},H_{ij}} > 0$ and $I_{V_{ij}} = I_{H_{ij}} = I_{V_{i+1j}} = I_{H_{ij+1}} = 1$, the other cases being similar and use $*$ to indicate the fluxes emerging from the node. For future use, we also compute the derivative of the solution with respect to permeability parameters.

If *RS*1 happens then the solution is given by $(O^*_{V_{ij}}, O^*_{H_{ij}}) = (\widehat{O}_{V_{ij}}, \widehat{O}_{H_{ij}})$. We compute:

$$\frac{\partial O^*_{V_{ij}}}{\partial \gamma_{V_{ij}}} = \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{H_{ij}}} = \widehat{Q}, \qquad \frac{\partial O^*_{V_{ij}}}{\partial \gamma_{H_{ij}}} = \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{V_{ij}}} = 0.$$

In the second case, *RS*2, the solution is given either by point B_1 or by point B_2, see Fig. 3. If B_1 gives the solution, then $B_1 = (O^*_{V_{ij}}, O^*_{H_{ij}}) = (\widehat{O}_{V_{ij}}, \widetilde{O}_{H_{ij}})$, where $\widetilde{O}_{H_{ij}} < \widehat{O}_{H_{ij}}$ and we get

$$\frac{\partial O^*_{V_{ij}}}{\partial \gamma^*_{V_{ij}}} = \widehat{Q}, \qquad \frac{\partial O^*_{V_{ij}}}{\partial \gamma_{H_{ij}}} = \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{H_{ij}}} = 0,$$

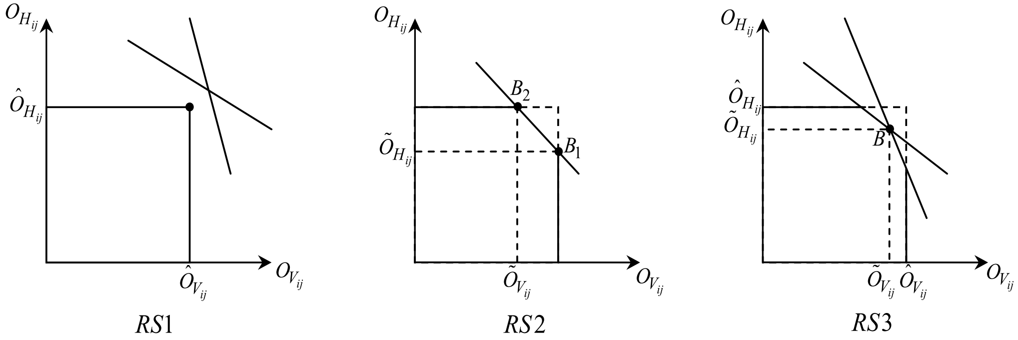

Fig. 3. Possible solvers, *RS*1, *RS*2, and *RS*3.

while

$$\frac{\partial O^*_{H_{ij}}}{\partial \gamma_{V_{ij}}} = -\frac{\alpha_{ij}}{\beta_{ij}}\widehat{Q} \quad \text{or} \quad \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{V_{ij}}} = -\frac{1-\alpha_{ij}}{1-\beta_{ij}}\widehat{Q},$$

depending on which line, defined by (7) or (8), intersects the rectangular region. If, instead, the solution is given by B_2, then $B_2 = (O^*_{V_{ij}}, O^*_{H_{ij}}) = (\widetilde{O}_{V_{ij}}, \widehat{O}_{H_{ij}})$, where $\widetilde{O}_{V_{ij}} < \widehat{O}_{V_{ij}}$ and we have

$$\frac{\partial O^*_{V_{ij}}}{\partial \gamma_{V_{ij}}} = \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{V_{ij}}} = 0, \qquad \frac{\partial O^*_{H_{ij}}}{\partial \gamma_{H_{ij}}} = \widehat{Q}.$$

Also

$$\frac{\partial O^*_{V_{ij}}}{\partial \gamma_{H_{ij}}} = -\frac{\beta_{ij}}{\alpha_{ij}}\widehat{Q} \quad \text{or} \quad \frac{\partial O^*_{V_{ij}}}{\partial \gamma_{H_{ij}}} = -\frac{1-\beta_{ij}}{1-\alpha_{ij}}\widehat{Q},$$

depending on which line, (7) or (8), intersects the rectangular region.

In the third case, *RS*3, the solution is given by $B = (O^*_{V_{ij}}, O^*_{H_{ij}}) = (\widetilde{O}_{V_{ij}}, \widetilde{O}_{H_{ij}})$, where $\widetilde{O}_{V_{ij}} < \widehat{O}_{V_{ij}}$ and $\widetilde{O}_{H_{ij}} < \widehat{O}_{H_{ij}}$. Moreover, the solution does not depend on the permeability parameters $\gamma_{V_{ij}}$ and $\gamma_{H_{ij}}$.

3.1.1. *Riemann Solver for the case* 1

Notice that, in case 1 (see Section 2.1.1), the 4-tuple $(A_{V_{i+1j}}, A_{H_{ij+1}}, O_{V_{ij}}, O_{H_{ij}})$, given by the Riemann Solver for the node $(i,j) \in \mathcal{J}$, is determined by: α_{ij}; β_{ij}; $\gamma_{H_{ij}}$; $\gamma_{V_{ij}}$; ΔN of roads connected to (i,j); delayed (A, O) for other nodes. In fact, from (2), $A_{V_{i+1j}}$ ($A_{H_{ij+1}}$ resp.) depends on delayed $O_{V_{i+1j}}$ ($O_{H_{ij+1}}$ resp.) if $\Delta N_{V_{i+1j}} = \Delta N^{\max}_{V_{i+1j}}$ ($\Delta N_{V_{i+1j}} = \Delta N^{\max}_{V_{i+1j}}$). From (3), $O_{V_{ij}}$ ($O_{H_{ij}}$ resp.) depends on delayed $A_{V_{ij}}$ ($A_{H_{ij}}$ resp.) if $\Delta N_{V_{ij}} = 0$ ($\Delta N_{H_{ij}} = 0$ resp.) and on $\gamma_{V_{ij}}\widehat{Q}$ ($\gamma_{H_{ij}}\widehat{Q}$ resp.) if $\Delta N_{V_{ij}} > 0$ ($\Delta N_{H_{ij}} > 0$).

3.1.2. *Riemann Solver for the case* 2

In the case 2 (see Section 2.1.2), the 4-tuple $(A_{V_{i+1j}}, A_{H_{ij+1}}, O_{V_{ij}}, O_{H_{ij}})$, determined by the Riemann Solver for the node $(i,j) \in \mathcal{J}$, depends on: α_{ij}; β_{ij}; $\gamma_{H_{ij}}$; $\gamma_{V_{ij}}$; l of roads connected to (i,j); delayed (A, O) for other nodes. In fact, from (2), $A_{V_{i+1j}}$ ($A_{H_{ij+1}}$ resp.) depends on delayed $O_{V_{i+1j}}$ ($O_{H_{ij+1}}$ resp.) if $l_{V_{i+1j}} < L_{V_{i+1j}}$ ($l_{H_{ij+1}} = L_{H_{ij+1}}$). From (3), $O_{V_{ij}}$ ($O_{H_{ij}}$ resp.) depends on delayed $A_{V_{ij}}$ ($A_{H_{ij}}$ resp.) if $\Delta N_{V_{ij}} = 0$ ($\Delta N_{H_{ij}} = 0$ resp.) and on $\gamma_{V_{ij}}\widehat{Q}$ ($\gamma_{H_{ij}}\widehat{Q}$ resp.) if $\Delta N_{V_{ij}} > 0$ ($\Delta N_{H_{ij}} > 0$).

Unlike the case 1, delays δ are state-dependent, i.e. $\delta = \delta(l)$.

4. Optimal control for networks

Let us now consider an optimal control problem for a Barcelona type network, representing the dynamics over the networks in the form of a control system:

$$\dot{x} = f(x, \gamma, \gamma_\delta), \tag{9}$$

where x (the number of delayed vehicles ΔN for case 1 and the lengths l of traffic jams in case 2) is the state, γ (the permeabilities) the control, while γ_δ represents delayed controls.

Given a class $\mathcal{U}$ of admissible controls and a fixed optimization horizon $[0, T]$, we can introduce the variable y_i satisfying $y_i(0) = 0$ and $\dot{y}_i = \Delta N_i(x, \gamma, \gamma_\delta)$, $i \in \mathcal{R}$, and state an optimal control problem as:

$$\min_{\gamma \in \mathcal{U}} \sum_{i \in \mathcal{R}} y_i(T),$$

for some fixed initial condition $\bar{x}$ at time 0. This corresponds to minimization of delayed vehicles over the whole network as function of the permeability parameters.

We now analyze in detail the dynamics, finding simple formulations for the case of not empty queues, i.e. $\Delta N_i > 0$ (resp. $l_i > 0$), while showing how the dynamics cannot be expressed in a simple way for the opposite case. More precisely, the evolution of each state variable depends on the whole networks via nested equations.

4.1. Not empty queues

4.1.1. Case 1

In this case, the dynamics for the whole network can be described by a general nonlinear control system of type (9), where $x = (y_{V_{ij}}, y_{H_{ij}}, \Delta N_{V_{ij}}, \Delta N_{H_{ij}})$. Fix a generic node $(i, j) \in \mathcal{J}$ of the network (of type (I)) and assume that queues on roads V_{ij} and H_{ij} are not zero, namely: $\Delta N_{V_{ij}} > 0$, $\Delta N_{H_{ij}} > 0$. Then we have that:

$$A_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{V_0}\right) = RS\left(\alpha_{i-1j}, \beta_{i-1j}, \gamma_{V_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right), \gamma_{H_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right)\right),$$

$$O_{V_{ij}}(t) = RS\big(\alpha_{ij}, \beta_{ij}, \gamma_{V_{ij}}(t), \gamma_{H_{ij}}(t)\big),$$

$$\begin{aligned}\Delta\dot{N}_{V_{ij}} &= A_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{V_0}\right) - O_{V_{ij}}(t) \\ &= RS\left(\gamma_{V_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right), \gamma_{H_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right)\right) - RS\big(\gamma_{V_{ij}}(t), \gamma_{H_{ij}}(t)\big),\end{aligned}$$

where, for simplicity, the dependence on traffic distribution coefficients, not dependent on time, was omitted in the last equation. Treating in a similar way the expression for the road H_{ij} (that can be obtained substituting H with V and i with j), we get the four equations:

$$\begin{aligned}\dot{y}_{V_{ij}} &= \Delta N_{V_{ij}}, \\ \dot{y}_{H_{ij}} &= \Delta N_{H_{ij}}, \\ \Delta\dot{N}_{V_{ij}} &= RS\left(\gamma_{V_{ij}}(t), \gamma_{H_{ij}}(t), \gamma_{V_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right), \gamma_{H_{i-1j}}\left(t - \frac{L_{V_{ij}}}{V_0}\right)\right), \\ \Delta\dot{N}_{H_{ij}} &= RS\left(\gamma_{V_{ij}}(t), \gamma_{H_{ij}}(t), \gamma_{V_{ij-1}}\left(t - \frac{L_{H_{ij}}}{V_0}\right), \gamma_{H_{ij-1}}\left(t - \frac{L_{H_{ij}}}{V_0}\right)\right).\end{aligned}$$

4.1.2. Case 2

Also in this case, the dynamics for the whole network can be described by a general nonlinear control system of type (9), where $x = (y_{V_{ij}}, y_{H_{ij}}, l_{V_{ij}}, l_{H_{ij}}, \Delta N_{V_{ij}}, \Delta N_{H_{ij}})$. Fix a node $(i, j) \in \mathcal{J}$ (of type (I)) and assume that $\Delta N_{V_{ij}}$ and $\Delta N_{H_{ij}}$ are both not zero. Then, we have that:

$$A_{V_{ij}}\left(t - \frac{L_{V_{ij}} - l_{V_{ij}}(t)}{V_0}\right) = RS\left(\gamma_{V_{i-1j}}\left(t - \frac{L_{V_{ij}} - l_{V_{ij}}(t)}{V_0}\right), \gamma_{H_{i-1j}}\left(t - \frac{L_{V_{ij}} - l_{V_{ij}}(t)}{V_0}\right)\right),$$
$$O_{V_{ij}}\left(t - \frac{l_{V_{ij}}(t)}{c}\right) = RS\left(\gamma_{V_{ij}}\left(t - \frac{l_{V_{ij}}(t)}{c}\right), \gamma_{H_{ij}}\left(t - \frac{l_{V_{ij}}(t)}{c}\right)\right),$$

hence,

$$\dot{l}_{V_{ij}}(t) = h_{V_{ij}}\left((\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\left(t - \frac{L_{V_{ij}} - l_{V_{ij}}(t)}{V_0}\right), (\gamma_{V_{ij}}, \gamma_{H_{ij}})\left(t - \frac{l_{V_{ij}}(t)}{c}\right)\right),$$
$$\dot{l}_{H_{ij}}(t) = h_{H_{ij}}\left((\gamma_{H_{ij-1}}, \gamma_{V_{ij-1}})\left(t - \frac{L_{H_{ij}} - l_{H_{ij}}(t)}{V_0}\right), (\gamma_{H_{ij}}, \gamma_{V_{ij}})\left(t - \frac{l_{H_{ij}}(t)}{c}\right)\right),$$

where $h_{V_{ij}}$ is a function of $A_{V_{ij}}$ and $O_{V_{ij}}$, while $h_{H_{ij}}$ is a function of $A_{H_{ij}}$ and $O_{H_{ij}}$, both defined via the Riemann Solver at node (i, j). Notice that, for simplicity, the dependence on traffic distribution coefficients, not dependent on time, was omitted in the previous equations. Considering similar expressions for the road H_{ij}, we have the equations:

$$\dot{y}_{V_{ij}} = \Delta N_{V_{ij}},$$
$$\dot{y}_{H_{ij}} = \Delta N_{H_{ij}},$$
$$\dot{l}_{V_{ij}} = h_{V_{ij}}\big((\gamma_{V_{ij}}, \gamma_{H_{ij}})\big(t - \delta_{3,V_{ij}}(t)\big), (\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\big(t - \delta_{1,V_{ij}} + \delta_{2,V_{ij}}(t)\big)\big),$$
$$\dot{l}_{H_{ij}} = h_{H_{ij}}\big((\gamma_{V_{ij}}, \gamma_{H_{ij}})\big(t - \delta_{3,H_{ij}}(t)\big), (\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\big(t - \delta_{1,H_{ij}} + \delta_{2,H_{ij}}(t)\big)\big),$$

where

$$\delta_{1,a} := \frac{L_a}{V_0}; \qquad \delta_{2,a}(t) := \frac{l_a(t)}{V_0}; \qquad \delta_{3,a}(t) := \frac{l_a(t)}{c}.$$

Remark 2. Notice that the previous equations contain three different types of delays in controls. Only the delay $\delta_{1,a}$ is constant with respect to time.

4.2. Empty queues and nested equations

4.2.1. Case 1

Assume that, for roads V_{ij}, V_{i-1j}, and V_{i-2j} (see Fig. 4), queues are zero, i.e. $\Delta N_{V_{ij}} = \Delta N_{V_{i-1j}} = \Delta N_{V_{i-2j}} = 0$. Consider the road V_{ij}, for which we have:

$$\dot{y}_{V_{ij}} = \Delta N_{V_{ij}}, \tag{10}$$

$$\Delta \dot{N}_{V_{ij}} = A_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{V_0}\right) - O_{V_{ij}}(t). \tag{11}$$

Notice that $A_{V_{ij}}(t)$ is determined by solving the dynamics at the node $(i - 1, j)$ and, in particular:

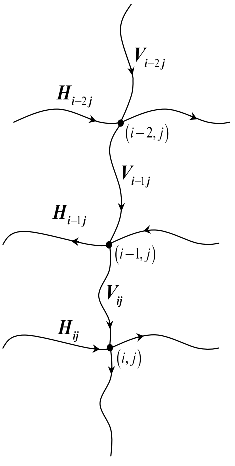

Fig. 4. Roads with zero queues.

$$A_{V_{ij}}(t) = RS\big(O_{V_{i-1j}}(t), O_{H_{i-1j}}(t), \gamma_{V_{i-1j}}(t), \gamma_{H_{i-1j}}(t)\big),$$

dropping, for simplicity, the dependence on $\alpha_{i-1j}, \beta_{i-1j}$. It follows that

$$A_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{V_0}\right) = RS\left((O_{V_{i-1j}}, O_{H_{i-1j}})\left(t - \frac{L_{V_{ij}}}{V_0}\right), (\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\left(t - \frac{L_{V_{ij}}}{V_0}\right)\right). \tag{12}$$

Since $\Delta N_{V_{i-1j}} = 0$,

$$O_{V_{i-1j}}(t) = g\left(\gamma_{V_{i-1j}}(t), A_{V_{i-1j}}\left(t - \frac{L_{V_{i-1j}}}{V_0}\right)\right),$$

where g is some function. Moreover, we have

$$A_{V_{i-1j}}\left(t - \frac{L_{V_{i-1j}}}{V_0}\right) = RS\left((O_{V_{i-2j}}, O_{H_{i-2j}})\left(t - \frac{L_{V_{i-1j}}}{V_0}\right), (\gamma_{V_{i-2j}}, \gamma_{H_{i-2j}})\left(t - \frac{L_{V_{i-1j}}}{V_0}\right)\right),$$

dropping, again for simplicity, the dependence on $\alpha_{i-2j}, \beta_{i-2j}$. Then Eq. (12) becomes:

$$A_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{V_0}\right) = RS\left(g(RS), (O_{H_{i-1j}}, \gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\left(t - \frac{L_{V_{ij}}}{V_0}\right)\right), \tag{13}$$

where

$$g(RS) = g\left(RS\left((O_{V_{i-2j}}, O_{H_{i-2j}}, \gamma_{V_{i-2j}}, \gamma_{H_{i-2j}})\left(t - \frac{L_{V_{i-1j}}}{V_0} - \frac{L_{V_{ij}}}{V_0}\right)\right)\right). \tag{14}$$

A similar result can be obtained for the road H_{ij}. It is then clear that solving dynamics for the node (i, j) implies the adoption of a "nested equation" (13)–(14), that indicates how phenomena at (i, j) are dependent on the dynamics on many other nodes. This in turn implies that the evolution of each state variable $y_{V_{ij}}$ and $\Delta N_{V_{ij}}$, given by (10)–(11), can be written only in terms of all nodes composing the networks.

4.2.2. Case 2

Assume again $\Delta N_{V_{ij}} = \Delta N_{V_{i-1j}} = \Delta N_{V_{i-2j}} = 0$ and, for simplicity drop the dependence on distribution coefficients. For road V_{ij}, we have

$$A_{V_{ij}}\big(t-\delta_{1,V_{ij}}+\delta_{2,V_{ij}}(t)\big) = RS\big((O_{V_{i-1j}}, O_{H_{i-1j}}, \gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\big(t-\delta_{1,V_{ij}}+\delta_{2,V_{ij}}(t)\big)\big).$$

Since $\Delta N_{V_{i-1j}} = 0$, we get:

$$O_{V_{i-1j}}(t) = g\big(\gamma_{V_{i-1j}}(t), A_{V_{ij}}\big(t-\delta_{1,V_{ij}}+\delta_{2,V_{ij}}(t)\big)\big),$$

where g is some function. Moreover,

$$A_{V_{i-1j}}\big(\varphi(t)\big) = RS\big((O_{V_{i-2j}}, O_{H_{i-2j}}, \gamma_{V_{i-2j}}, \gamma_{H_{i-2j}})\big(\varphi(t)\big)\big),$$

where $\varphi(t) = t-\delta_{1,V_{i-1j}}+\delta_{2,V_{i-1j}}(t)$. Hence, we obtain that

$$A_{V_{ij}}\big(\varphi(t)\big) = RS\big(g(RS), (O_{H_{i-1j}}, \gamma_{V_{i-1j}}, \gamma_{H_{i-1j}})\big(\varphi(t)\big)\big), \tag{15}$$

where $g(RS) = g(RS((O_{V_{i-2j}}, O_{H_{i-2j}}, \gamma_{H_{i-2j}})(t')))$, with $t' = t-\delta_{1,V_{i-1j}}+\delta_{2,V_{i-1j}}(t)-\delta_{1,V_{ij}}+\delta_{2,V_{ij}}(t)$. A similar result can be obtained for the road H_{ij}. Again we are in presence of nested equations involving the whole network.

4.3. A hybrid dynamic

In this section we introduce a hybrid dynamic to avoid the nested equations showing up in case of empty queues. We can replace continuous equations involving the whole network at the price of introducing some extra discrete (logic) variables. The latter, in turn, are affected and affect the continuous variables evolution.

4.3.1. Case 1

Define the logic variables $\varepsilon_{V_{ij}}$ as follows:

$$\varepsilon_{V_{ij}} := \begin{cases} -1 & \text{if } \Delta N_{V_{ij}} = 0, \\ 0 & \text{if } 0 < \Delta N_{V_{ij}} < \Delta N^{\max}_{V_{ij}}, \\ +1 & \text{if } \Delta N_{V_{ij}} = \Delta N^{\max}_{V_{ij}}. \end{cases}$$

For $\varepsilon_{H_{ij}}$, the definition is similar, substituting V with H. A complete hybrid dynamic for the node (i,j) is then given by the following equations (for simplicity, we drop the dependence on distribution coefficient and on time, using the exponent δ to indicate a delayed dependence on time):

$$\begin{gathered}
\dot{y}_{V_{ij}} = \Delta N_{V_{ij}}, \qquad \dot{y}_{H_{ij}} = \Delta N_{H_{ij}}, \\
\Delta\dot{N}_{V_{ij}} = A^{\delta}_{V_{ij}} - O_{V_{ij}}, \qquad \Delta\dot{N}_{H_{ij}} = A^{\delta}_{H_{ij}} - O_{H_{ij}}, \\
A_{V_{ij}} = RS\big(\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}}, O^{\delta}_{V_{ij}}, O^{\delta}_{H_{i-1j+1}}, A^{\delta}_{V_{i-1j}}, A^{\delta}_{H_{i-1j}}, \varepsilon_{V_{i-1j}}, \varepsilon_{H_{i-1j}}, \varepsilon_{V_{ij}}, \varepsilon_{H_{i-1j+1}}\big), \\
O_{V_{ij}} = RS\big(\gamma_{V_{ij}}, \gamma_{H_{ij}}, A^{\delta}_{V_{ij}}, A^{\delta}_{H_{ij}}, O^{\delta}_{V_{i+1j}}, O^{\delta}_{H_{ij+1}}, \varepsilon_{V_{ij}}, \varepsilon_{H_{ij}}, \varepsilon_{V_{i+1j}}, \varepsilon_{H_{ij+1}}\big), \\
A_{H_{ij}} = RS\big(\gamma_{V_{ij-1}}, \gamma_{H_{ij-1}}, O^{\delta}_{V_{i+1j-1}}, O^{\delta}_{H_{ij}}, A^{\delta}_{V_{ij-1}}, A^{\delta}_{H_{ij-1}}, \varepsilon_{V_{ij-1}}, \varepsilon_{V_{i+1j-1}}, \varepsilon_{H_{ij}}, \varepsilon_{H_{ij-1}}\big), \\
O_{H_{ij}} = RS\big(\gamma_{V_{ij}}, \gamma_{H_{ij}}, A^{\delta}_{V_{ij}}, A^{\delta}_{H_{ij}}, O^{\delta}_{V_{i+1j}}, O^{\delta}_{H_{ij+1}}, \varepsilon_{V_{ij}}, \varepsilon_{H_{ij}}, \varepsilon_{V_{i+1j}}, \varepsilon_{H_{ij+1}}\big).
\end{gathered}$$

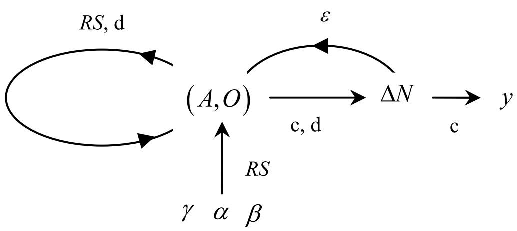

Fig. 5. Scheme of the hybrid dynamics for case 1.

The dynamic of control parameters γ (and distribution coefficients α or β) influence the evolution of the couple (A, O) through *RS*. In turn, the values of (A, O) influence themselves through *RS* and determine the continuous dynamics of ΔN. The dynamics of ΔN defines that of y and discrete changes, through ε, of the couple (A, O). A summarizing scheme for this hybrid dynamics is given in Fig. 5, where c, resp. d, indicates if the dynamic is continuous, resp. delayed.

4.3.2. Case 2

Define the logic variables $\varepsilon^l_{V_{ij}}$ and $\varepsilon^{\Delta N}_{V_{ij}}$ as follows:

$$\varepsilon^l_{V_{ij}} := \begin{cases} 0 & \text{if } l_{V_{ij}} < L_{V_{ij}}, \\ +1 & \text{if } l_{V_{ij}} = L_{V_{ij}}, \end{cases}$$

$$\varepsilon^{\Delta N}_{V_{ij}} := \begin{cases} 0 & \text{if } \Delta N_{V_{ij}} = 0, \\ +1 & \text{if } \Delta N_{V_{ij}} > 0. \end{cases}$$

For $\varepsilon^l_{H_{ij}}$ and $\varepsilon^{\Delta N}_{H_{ij}}$, the definition is similar, if we substitute V with H. The following equations describe a hybrid dynamic for the node (i, j) (again, for simplicity, the dependence on distribution coefficients and time is omitted, the exponent $\delta(l)$ indicates a state-dependent delay in time, while h is a function depending on the Riemann Solver):

$$\dot{y}_{V_{ij}} = \Delta N_{V_{ij}}, \qquad \dot{y}_{H_{ij}} = \Delta N_{H_{ij}},$$

$$\dot{l}_{V_{ij}} = h_{V_{ij}}\big(A^{\delta(l)}_{V_{ij}}, O^{\delta(l)}_{V_{ij}}\big), \qquad \dot{l}_{H_{ij}} = h_{H_{ij}}\big(A^{\delta(l)}_{H_{ij}}, O^{\delta(l)}_{H_{ij}}\big),$$

$$A_{V_{ij}} = RS\big(\gamma_{V_{i-1j}}, \gamma_{H_{i-1j}}, O_{V_{ij}}, O_{H_{i-1j+1}}, A^{\delta(l)}_{V_{i-1j}}, A^{\delta(l)}_{H_{i-1j}}, \varepsilon^l_{V_{i-1j}}, \varepsilon^l_{H_{i-1j}}, \varepsilon^l_{V_{ij}}, \varepsilon^l_{H_{i-1j+1}}\big),$$

$$O_{V_{ij}} = RS\big(\gamma_{V_{ij}}, \gamma_{H_{ij}}, A^{\delta(l)}_{V_{ij}}, A^{\delta(l)}_{H_{ij}}, O^{\delta(l)}_{V_{i+1j}}, O^{\delta(l)}_{H_{ij+1}}, \varepsilon^{\Delta N}_{V_{ij}}, \varepsilon^{\Delta N}_{H_{ij}}, \varepsilon^{\Delta N}_{V_{i+1j}}, \varepsilon^{\Delta N}_{H_{ij+1}}\big),$$

$$A_{H_{ij}} = RS\big(\gamma_{V_{ij-1}}, \gamma_{H_{ij-1}}, O^{\delta(l)}_{V_{i+1j-1}}, O^{\delta(l)}_{H_{ij}}, A^{\delta(l)}_{V_{ij-1}}, A^{\delta(l)}_{H_{ij-1}}, \varepsilon^l_{V_{ij-1}}, \varepsilon^l_{V_{i+1j-1}} \varepsilon^l_{H_{ij}}, \varepsilon^l_{H_{ij-1}}\big),$$

$$O_{H_{ij}} = RS\big(\gamma_{V_{ij}}, \gamma_{H_{ij}}, A^{\delta(l)}_{V_{ij}}, A^{\delta(l)}_{H_{ij}}, O^{\delta(l)}_{V_{i+1j}}, O^{\delta(l)}_{H_{ij+1}}, \varepsilon^{\Delta N}_{V_{ij}}, \varepsilon^{\Delta N}_{H_{ij}}, \varepsilon^{\Delta N}_{V_{i+1j}}, \varepsilon^{\Delta N}_{H_{ij+1}}\big).$$

The dynamics of controls γ (and distribution coefficients α and β) influence the values of the couple (A, O) through *RS*, which in turn affect themselves, and the dynamics of l, in a delayed way, and that of ΔN. The dynamics of ΔN determines that of y, and, through the logic variables $\varepsilon^{\Delta N}$, the values of (A, O). Finally, l influences, through the logical variables ε^l, the values of (A, O). A summarizing scheme for this hybrid dynamics is in Fig. 6, where c, resp. d, indicates if the dynamic is continuous, resp. delayed.

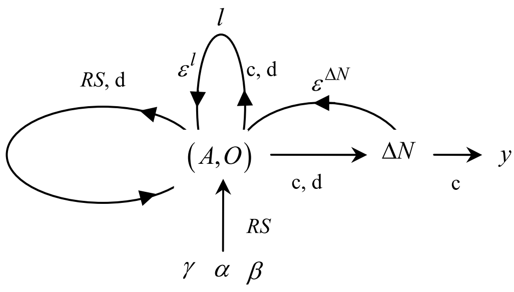

Fig. 6. Scheme of the hybrid dynamics for case 2.

5. Needle variations and variational equations

In what follows, we analyze the sensitivity of the system defined by case 1, see Section 2.1.1, with respect to control variations. To achieve this goal, we adopt the point of view of Pontryagin Maximum Principle (PMP), see [1], and consider special control variations, called needle variations. The latter give rise to variational equations along trajectories to determine the relative effects on the dynamics. Let us start recalling basic definitions.

Consider an optimal control problem for a control system of the following type:

$$\dot{x} = f(x, \gamma, \gamma_\delta),$$

where $\gamma_\delta(t) = \gamma(t-\delta)$ and $\delta > 0$. Fix a candidate optimal control $\mathcal{U} \ni \gamma^* : [0,T] \mapsto U = [0,1]$ and let x^* be the corresponding trajectory, starting from a given point $\bar{x}$. A needle variation is defined as follows:

Definition 3 *(Needle variation).* Consider the map $\varphi : t \mapsto f(x^*(t), \gamma^*(t), \gamma^*_\delta(t))$ and let τ be a Lebesgue point for φ. Given $\omega \in U$, define a family of controls $\eta_\gamma(t, \tau, \zeta, \omega)$, $\zeta \in [0, \tau[$, in the following way:

$$\eta_\gamma(t,\tau,\zeta,\omega) := \begin{cases} \gamma^*(t) & t \in [0, \tau - \zeta[, \\ \omega & t \in [\tau-\zeta, \tau[, \\ \gamma^*(t) & t \in [\tau, T], \end{cases}$$

and let $\eta_x(t,\tau,\zeta,\omega)$ be the trajectories corresponding to η_γ with $\eta_x(0,\tau,\zeta,\omega) = \bar{x}$. We call the couple $(\eta_\gamma, \eta_x) = (\eta_\gamma, \eta_x)(\tau, \omega)$ a needle variation of $(x^*, \gamma^*, \gamma^*_\delta)$. If the trajectories are uniquely determined by controls we use the simplified notation $\eta_\gamma(\tau, \omega)$.

If the cost is given as in previous section, for γ^* to be optimal we need:

$$\nabla\left(\sum_{i\in\mathcal{R}} y_i^*(T)\right) \cdot v(T) \geqslant 0,$$

where

$$v(t) = \frac{d}{d\zeta}\eta_x(t,\tau,\zeta,\omega)\bigg|_{\zeta=0}$$

is the tangent vector to the curve $\zeta \mapsto \eta_x(T, \tau, \zeta, \omega)$ at $\zeta = 0$. The vector v, for $t > \tau$, satisfies the variational equation:

$$\dot{v} = D_x f\left(x^*, \gamma^*, \gamma^*_\delta\right) \cdot v,$$

with initial condition:

$$v(\tau) = f\big(x^*(\tau), \omega, \gamma_\delta^*(\tau)\big) - f\big(x^*(\tau), \gamma^*(\tau), \gamma_\delta^*(\tau)\big)$$

and presents a jump at time $\tau + \delta$ given by

$$v\big((\tau+\delta)^+\big) = v\big((\tau+\delta)^-\big) + f\big(x^*(\tau+\delta), \gamma^*(\tau+\delta), \omega\big) - f\big(x^*(\tau+\delta), \gamma^*(\tau+\delta), \gamma^*(\tau)\big).$$

Recalling the notation for a Barcelona network, if a variation of $\gamma_{V_{ij}}$ occurs at node (i,j), we have to consider the tangent vectors $v_{V_{ij}}^y$ and $v_{V_{ij}}^{\Delta N}$ for the variables $y_{V_{ij}}$ and $\Delta N_{V_{ij}}$, respectively. Similarly for H_{ij}. Hence, the variational equations are described by

$$\dot v_{V_{ij}}^y = v_{V_{ij}}^{\Delta N}, \qquad \dot v_{H_{ij}}^y = v_{H_{ij}}^{\Delta N}, \qquad \dot v_{V_{ij}}^{\Delta N} = \dot v_{H_{ij}}^{\Delta N} = 0.$$

To study the effect of needle variations on continuous and logic variables, we need some further notation. Similarly to Definition 3, we indicate by $\eta_A(\tau, \overline{A})$ the needle variation of A at time τ and with value $\overline{A}$; $\eta_O(\tau, \overline{O})$ the needle variation of O at time τ and with value $\overline{O}$. We also use the symbol $\eta_\varepsilon = \eta_\varepsilon(\tau, \varepsilon_1, \varepsilon_2)$ to indicate a *switching variation* of an evolution of ε which, at time τ, switches from ε_1 to ε_2. More precisely, $\eta_\varepsilon(t, \tau, \zeta, \varepsilon_1, \varepsilon_2)$ switches from ε_1 to ε_2 at time $\tau - \zeta$.

To fix the ideas, we focus on needle variations of a single control, thus on the two following cases:

A1 no variation for $\gamma_{H_{ij}}^*$ occurs; a needle variation at time $\tau \in]0, T]$ with value $\omega_{V_{ij}} \in [0,1]$ occurs for $\gamma_{V_{ij}}(t)$;

A2 no variation for $\gamma_{V_{ij}}^*$ occurs; a needle variation at time $\tau \in]0, T]$ with value $\omega_{H_{ij}} \in [0,1]$ occurs for $\gamma_{H_{ij}}(t)$.

Needle variations of permeability parameters (controls) generate other needle variations for the arrival and departure flows, which in turn provoke jumps in the variational vectors for delayed vehicles. We provide now some results to show the cascade of effects generated by a needle variation of controls.

Proposition 4. *Consider a node (i,j). A needle variation $\eta_{\gamma_{V_{ij}}}(\tau, \omega)$ or a needle variation $\eta_{\gamma_{H_{ij}}}(\tau, \omega)$ generates needle variations $\eta_{O_{V_{ij}}}(\tau, \overline{O}_{V_{ij}})$, $\eta_{O_{H_{ij}}}(\tau, \overline{O}_{H_{ij}})$, $\eta_{A_{V_{i+1j}}}(\tau, \overline{A}_{V_{i+1j}})$, and $\eta_{A_{H_{ij+1}}}(\tau, \overline{A}_{H_{ij+1}})$, where $\overline{O}_{V_{ij}}$, $\overline{O}_{H_{ij}}$, $\overline{A}_{V_{i+1j}}$, and $\overline{A}_{H_{ij+1}}$ are determined by RS at node (i,j).*

Proof. Fix a node (i,j), assume that $\Delta N_{V_{ij}, H_{ij}} > 0$ and the number of lanes for each road is one, namely $I_{V_{ij}} = I_{H_{ij}} = I_{H_{ij+1}} = I_{V_{i+1j}} = 1$. In what follows, for A, O and γ we drop the dependence on time. Three possible cases for *RS* (namely *RS*1, *RS*2, and *RS*3) at node (i,j) can occur, see Section 3.1. If *RS*1 happens, then the solution in the $(O_{V_{ij}}, O_{H_{ij}})$ plane is given by

$$O_{V_{ij}}^* = \gamma_{V_{ij}}^* \widehat{Q}, \tag{16}$$

$$O_{H_{ij}}^* = \gamma_{H_{ij}}^* \widehat{Q}, \tag{17}$$

and the corresponding solutions for outgoing roads are:

$$A_{H_{ij+1}}^* = \alpha_{ij}\gamma_{V_{ij}}^* \widehat{Q} + \beta_{ij}\gamma_{H_{ij}}^* \widehat{Q}, \qquad A_{V_{i+1j}}^* = (1-\alpha_{ij})\gamma_{V_{ij}}^* \widehat{Q} + (1-\beta_{ij})\gamma_{H_{ij}}^* \widehat{Q}. \tag{18}$$

Consider first case **A1**. From (16), the needle variation in $\gamma_{V_{ij}}$ provokes a needle variation in $O_{V_{ij}}$ given by

$$\eta_{O_{V_{ij}}}(t,\tau,\zeta,\overline{O}_{V_{ij}}) = \begin{cases} \overline{O}_{V_{ij}} & t\in[\tau-\zeta,\tau[,\\ \gamma^*_{V_{ij}}\widehat{Q} & \text{otherwise,}\end{cases}$$

where $\overline{O}_{V_{ij}} = \omega_{V_{ij}}\widehat{Q}$. As $\frac{\partial O_{H_{ij}}}{\partial \gamma_{V_{ij}}} = 0$, $O_{H_{ij}}$ presents no variation. On the other side, from (18), there are needle variations on outgoing flows given by

$$\eta_{A_{H_{ij+1}}}(t,\tau,\zeta,\overline{A}_{H_{ij+1}}) = \begin{cases} \overline{A}_{H_{ij+1}} & t\in[\tau-\zeta,\tau[,\\ \alpha_{ij}\gamma^*_{V_{ij}}\widehat{Q}+\beta_{ij}\gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise,}\end{cases} \tag{19}$$

$$\eta_{A_{V_{i+1j}}}(t,\tau,\zeta,\overline{A}_{V_{i+1j}}) = \begin{cases} \overline{A}_{V_{i+1j}} & t\in[\tau-\zeta,\tau[,\\ (1-\alpha_{ij})\gamma^*_{V_{ij}}\widehat{Q}+(1-\beta_{ij})\gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise,}\end{cases} \tag{20}$$

where

$$\overline{A}_{H_{ij+1}} = \alpha_{ij}\omega_{V_{ij}}\widehat{Q}+\beta_{ij}\gamma^*_{H_{ij}}\widehat{Q}, \qquad \overline{A}_{V_{i+1j}} = (1-\alpha_{ij})\omega_{V_{ij}}\widehat{Q}+(1-\beta_{ij})\gamma^*_{H_{ij}}\widehat{Q}.$$

Consider now case **A2**. Reasoning as before, as $\frac{\partial O_{V_{ij}}}{\partial \gamma_{H_{ij}}} = 0$, $O_{V_{ij}}$ has no variation, while, from (17), there is a variation in $O_{H_{ij}}$ given by

$$\eta_{O_{H_{ij}}}(t,\tau,\zeta,\overline{O}_{H_{ij}}) = \begin{cases} \overline{O}_{H_{ij}} & t\in[\tau-\zeta,\tau[,\\ \gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise,}\end{cases}$$

where $\overline{O}_{H_{ij}} = \omega_{H_{ij}}\widehat{Q}$. Needle variations in outgoing flows are as in (19) and (20) with the values of $\overline{A}_{H_{ij+1}}$ and $\overline{A}_{V_{i+1j}}$ equal to

$$\overline{A}_{H_{ij+1}} = \alpha_{ij}\gamma^*_{V_{ij}}\widehat{Q}+\beta_{ij}\omega_{H_{ij}}\widehat{Q}, \qquad \overline{A}_{V_{i+1j}} = (1-\alpha_{ij})\gamma^*_{V_{ij}}\widehat{Q}+(1-\beta_{ij})\omega_{H_{ij}}\widehat{Q}.$$

As for *RS*2, the solution, indicated by $(O^*_{V_{ij}}, O^*_{H_{ij}})$, is either given by

$$\left(O^*_{V_{ij}}, O^*_{H_{ij}}\right) = \left(\gamma^*_{V_{ij}}\widehat{Q}, \frac{\widehat{A}-a_{ij}\gamma^*_{V_{ij}}\widehat{Q}}{b_{ij}}\right), \tag{21}$$

or by

$$\left(O^*_{V_{ij}}, O^*_{H_{ij}}\right) = \left(\frac{\widehat{A}-b_{ij}\gamma^*_{H_{ij}}\widehat{Q}}{a_{ij}}, \gamma^*_{H_{ij}}\widehat{Q}\right), \tag{22}$$

while

$$A^*_{H_{ij+1}} = \alpha_{ij}O^*_{V_{ij}}+\beta_{ij}O^*_{H_{ij}}, \qquad A^*_{V_{i+1j}} = (1-\alpha_{ij})O^*_{V_{ij}}+(1-\beta_{ij})O^*_{H_{ij}}, \tag{23}$$

where a_{ij} (resp. b_{ij}) is either equal to α_{ij} (resp. β_{ij}) or to $1-\alpha_{ij}$ (resp. $1-\beta_{ij}$), depending on which line, between $\alpha_{ij}O_{V_{ij}}+\beta_{ij}O_{H_{ij}} = \widehat{A}_{H_{ij+1}}$ and $(1-\alpha_{ij})O_{V_{ij}}+(1-\beta_{ij})O_{H_{ij}} = \widehat{A}_{V_{i+1j}}$, intersects the convex region of possible fluxes in the $(O_{V_{ij}}, O_{H_{ij}})$ plane. Notice that $\widehat{A} = \widehat{A}_{H_{ij+1}}$ if $a_{ij} = \alpha_{ij}$ and $b_{ij} = \beta_{ij}$, while $\widehat{A} = \widehat{A}_{V_{i+1j}}$ in the opposite case.

Consider first case **A1**. If $(O^*_{V_{ij}}, O^*_{H_{ij}})$ is defined by (21), then:

$$\eta_{O_{V_{ij}}} = \begin{cases} \overline{O}_{V_{ij}} & t \in [\tau - \zeta, \tau[, \\ \gamma^*_{V_{ij}} \widehat{Q} & \text{otherwise}, \end{cases} \qquad \eta_{O_{H_{ij}}} = \begin{cases} \overline{O}_{H_{ij}} & t \in [\tau - \zeta, \tau[, \\ \frac{\widehat{A} - a_{ij}\gamma^*_{V_{ij}}\widehat{Q}}{b_{ij}} & \text{otherwise}, \end{cases}$$

where $\overline{O}_{V_{ij}} = \omega_{V_{ij}} \widehat{Q}$ and $\overline{O}_{H_{ij}} = (\widehat{A} - a_{ij}\omega_{V_{ij}}\widehat{Q})/b_{ij}$. Moreover, from (23), we get:

$$\eta_{A_{H_{ij+1}}}(t, \tau, \zeta, \overline{A}_{H_{ij+1}}) = \begin{cases} \overline{A}_{H_{ij+1}} & t \in [\tau - \zeta, \tau[, \\ \alpha_{ij}\gamma^*_{V_{ij}}\widehat{Q} + \beta_{ij}\frac{\widehat{A} - a_{ij}\gamma^*_{V_{ij}}\widehat{Q}}{b_{ij}} & \text{otherwise}, \end{cases}$$

$$\eta_{A_{V_{i+1j}}}(t, \tau, \zeta, \overline{A}_{V_{i+1j}}) = \begin{cases} \overline{A}_{V_{i+1j}} & t \in [\tau - \zeta, \tau[, \\ (1 - \alpha_{ij})\gamma^*_{V_{ij}}\widehat{Q} + (1 - \beta_{ij})\frac{\widehat{A} - a_{ij}\gamma^*_{V_{ij}}\widehat{Q}}{b_{ij}} & \text{otherwise}, \end{cases}$$

where

$$\overline{A}_{H_{ij+1}} = \alpha_{ij}\omega_{V_{ij}}\widehat{Q} + \beta_{ij}\frac{\widehat{A} - a_{ij}\omega_{V_{ij}}\widehat{Q}}{b_{ij}}, \qquad \overline{A}_{V_{i+1j}} = (1 - \alpha_{ij})\omega_{V_{ij}}\widehat{Q} + (1 - \beta_{ij})\frac{\widehat{A} - a_{ij}\omega_{V_{ij}}\widehat{Q}}{b_{ij}}.$$

If $(O^*_{V_{ij}}, O^*_{H_{ij}})$ is defined by (22), as $\frac{\partial O_{V_{ij}}}{\partial \gamma_{V_{ij}}} = \frac{\partial O_{H_{ij}}}{\partial \gamma_{V_{ij}}} = 0$, $O_{V_{ij}}$ and $O_{H_{ij}}$ have no variation and, from (23), also $A_{H_{ij+1}}$ and $A_{V_{i+1j}}$ remain unchanged.

Consider now case **A2**. If $(O^*_{V_{ij}}, O^*_{H_{ij}})$ is defined by (21), we have no variation for $O_{V_{ij}}$, $O_{H_{ij}}$, $A_{H_{ij+1}}$ and $A_{V_{i+1j}}$. If $(O^*_{V_{ij}}, O^*_{H_{ij}})$ is defined by (22), then:

$$\eta_{O_{V_{ij}}} = \begin{cases} \overline{O}_{V_{ij}} & t \in [\tau - \zeta, \tau[, \\ \frac{\widehat{A} - b_{ij}\gamma^*_{H_{ij}}\widehat{Q}}{a_{ij}} & \text{otherwise}, \end{cases} \qquad \eta_{O_{H_{ij}}} = \begin{cases} \overline{O}_{H_{ij}} & t \in [\tau - \zeta, \tau[, \\ \gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise}, \end{cases}$$

where $\overline{O}_{V_{ij}} = (\widehat{A} - b_{ij}\omega_{H_{ij}}\widehat{Q})/a_{ij}$ and $\overline{O}_{H_{ij}} = \omega_{H_{ij}}\widehat{Q}$. From (23), we have:

$$\eta_{A_{H_{ij+1}}}(t, \tau, \zeta, \overline{A}_{H_{ij+1}}) = \begin{cases} \overline{A}_{H_{ij+1}} & t \in [\tau - \zeta, \tau[, \\ \alpha_{ij}\frac{\widehat{A} - b_{ij}\gamma^*_{H_{ij}}\widehat{Q}}{a_{ij}} + \beta_{ij}\gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise}, \end{cases}$$

$$\eta_{A_{V_{i+1j}}}(t, \tau, \zeta, \overline{A}_{V_{i+1j}}) = \begin{cases} \overline{A}_{V_{i+1j}} & t \in [\tau - \zeta, \tau[, \\ (1 - \alpha_{ij})\frac{\widehat{A} - b_{ij}\gamma^*_{H_{ij}}\widehat{Q}}{a_{ij}} + (1 - \beta_{ij})\gamma^*_{H_{ij}}\widehat{Q} & \text{otherwise}, \end{cases}$$

where

$$\overline{A}_{H_{ij+1}} = \alpha_{ij}\frac{\widehat{A} - b_{ij}\omega_{H_{ij}}\widehat{Q}}{a_{ij}} + \beta_{ij}\omega_{H_{ij}}\widehat{Q}, \qquad \overline{A}_{V_{i+1j}} = (1 - \alpha_{ij})\frac{\widehat{A} - b_{ij}\omega_{H_{ij}}\widehat{Q}}{a_{ij}} + (1 - \beta_{ij})\omega_{H_{ij}}\widehat{Q}.$$

Consider finally case *RS*3, for which the solution in the $(O_{V_{ij}}, O_{H_{ij}})$ plane is:

$$O^*_{V_{ij}} = \frac{\beta_{ij}(\widehat{A}_{H_{ij+1}} + \widehat{A}_{V_{i+1j}}) - \widehat{A}_{H_{ij+1}}}{\beta_{ij} - \alpha_{ij}}, \qquad O^*_{H_{ij}} = \frac{(1-\alpha_{ij})\widehat{A}_{H_{ij+1}} - \alpha_{ij}\widehat{A}_{V_{i+1j}}}{\beta_{ij} - \alpha_{ij}}, \tag{24}$$

and the corresponding solutions on outgoing roads are:

$$A^*_{H_{ij+1}} = \alpha_{ij}\frac{\beta_{ij}(\widehat{A}_{H_{ij+1}} + \widehat{A}_{V_{i+1j}}) - \widehat{A}_{H_{ij+1}}}{\beta_{ij} - \alpha_{ij}} + \beta_{ij}\frac{(1-\alpha_{ij})\widehat{A}_{H_{ij+1}} - \alpha_{ij}\widehat{A}_{V_{i+1j}}}{\beta_{ij} - \alpha_{ij}}, \tag{25}$$

$$A^*_{V_{i+1j}} = (1-\alpha_{ij})\frac{\beta_{ij}(\widehat{A}_{H_{ij+1}} + \widehat{A}_{V_{i+1j}}) - \widehat{A}_{H_{ij+1}}}{\beta_{ij} - \alpha_{ij}} + (1-\beta_{ij})\frac{(1-\alpha_{ij})\widehat{A}_{H_{ij+1}} - \alpha_{ij}\widehat{A}_{V_{i+1j}}}{\beta_{ij} - \alpha_{ij}}. \tag{26}$$

Formulas (24), (25), and (26) indicate that no variations occur. □

Next result shows the effect of needle variations on outgoing flows.

Proposition 5. *A needle variation $\eta_{O_{V_{ij}}}(\tau, \overline{O}_{V_{ij}})$ determines a jump of $v^{\Delta N}_{V_{ij}}$ at time τ and, if $\Delta N_{V_{ij}}(\tau) = \Delta N^{\max}_{V_{ij}}$, it provokes the following needle variations: $\eta_{O_{V_{i-1j}}}(\tau+\delta, \overline{O}_{V_{i-1j}})$, $\eta_{O_{H_{i-1j}}}(\tau+\delta, \overline{O}_{H_{i-1j}})$, $\eta_{A_{V_{ij}}}(\tau+\delta, \overline{A}_{V_{ij}})$, $\eta_{A_{H_{i-1j+1}}}(\tau+\delta, \overline{A}_{H_{i-1j+1}})$, where $\overline{A}_{V_{ij}}$, $\overline{A}_{H_{i-1j+1}}$, $\overline{O}_{V_{i-1j}}$, $\overline{O}_{H_{i-1j}}$ are determined by RS at node $(i-1, j)$.*

Proof. As before, for A, O and γ we drop the dependence on time and assume that the number of lanes for each road of node $(i-1, j)$ is one, namely $I_{V_{i-1j}} = I_{H_{i-1j}} = I_{H_{i-1j+1}} = I_{V_{ij}} = 1$. If a needle variation $\eta_{O_{V_{ij}}}(t, \tau, \zeta, \overline{O}_{V_{ij}})$ occurs, then indicating by an exponent the dependence on ζ we have:

$$\Delta N^{\zeta}_{V_{ij}}(\tau) = \Delta N^*_{V_{ij}}(\tau) + \zeta\big(\overline{O}_{V_{ij}} - O^*_{V_{ij}}\big), \tag{27}$$

as long as the right-hand side is in the interval $[0, \Delta N^{\max}_{V_{ij}}]$. Thus, in the latter case, the tangent vector $v^{\Delta N}_{V_{ij}}$ satisfies:

$$v^{\Delta N}_{V_{ij}}(\tau) = \overline{O}_{V_{ij}} - O^*_{V_{ij}},$$

while $v_{V_{ij}}(\tau-) = 0$. Therefore, a needle variation at time τ for $O_{V_{ij}}$ implies a jump for $v^{\Delta N}_{V_{ij}}$ at time τ. In other words a needle variation of $\Delta N_{V_{ij}}$ occurs. However, if the right-hand side of (27) is not in the interval $[0, \Delta N^{\max}_{V_{ij}}]$, then such jump may be smaller or even vanish.

To determine the other variations, two cases must be considered:

(A) $0 \leqslant \Delta N_{V_{ij}}(\tau) < \Delta N^{\max}_{V_{ij}}$;
(B) $\Delta N_{V_{ij}}(\tau) = \Delta N^{\max}_{V_{ij}}$.

Assume first case **A** occurs. Since $\frac{\partial A_{V_{ij}}}{\partial O_{V_{ij}}} = 0$, the dynamics at node $(i-1, j)$ is not influenced by variations at node (i, j).

Consider now case **B**. As $\omega_{V_{ij}}$ is the value of the needle variation of $\gamma_{V_{ij}}$ at time τ, from (2) we have that $A_{V_{ij}}(\tau+\delta) = \omega_{V_{ij}}\widehat{Q}$, and a needle variation for $A_{V_{ij}}$ occurs at time $\tau+\delta$ given by

$$\eta_{A_{V_{ij}}}(t, \tau+\delta, \zeta, \overline{A}_{V_{ij}}) = \begin{cases} \overline{A}_{V_{ij}} & t \in [\tau+\delta-\zeta, \tau+\delta[, \\ A^*_{V_{ij}} & \text{otherwise}, \end{cases} \tag{28}$$

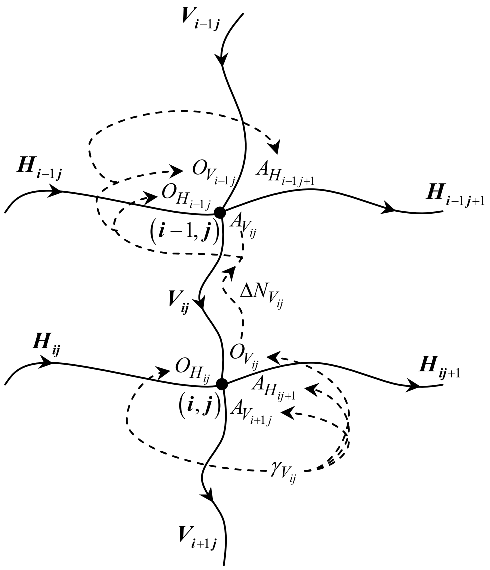

Fig. 7. Influence of a needle variation of $\gamma_{V_{ij}}$ on nodes (i,j) and $(i-1,j)$. Notice that $A_{V_{ij}}$ is affected, and hence dynamics at node $(i-1,j)$, only if $\Delta N_{V_{ij}} = \Delta N_{V_{ij}}^{\max}$.

where $\overline{A}_{V_{ij}} = \omega_{V_{ij}} \widehat{Q}$. This variation in turn implies needle variations of $O_{V_{i-1j}}$, $O_{H_{i-1j}}$ and $A_{H_{i-1j+1}}$ at time $\tau+\delta$. $\square$

With proofs similar to that of Proposition 5, we get the following:

Proposition 6. *A needle variation $\eta_{O_{H_{ij}}}(\tau, \overline{O}_{H_{ij}})$ determines a jump of $v_{H_{ij}}^{\Delta N}$ and, if $\Delta N_{H_{ij}} = \Delta N_{H_{ij}}^{\max}$, the following needle variations: $\eta_{O_{H_{ij-1}}}(\tau+\delta, \overline{O}_{H_{ij-1}})$, $\eta_{O_{V_{ij-1}}}(\tau+\delta, \overline{O}_{V_{ij-1}})$, $\eta_{A_{H_{ij}}}(\tau+\delta, \overline{A}_{H_{ij}})$, $\eta_{A_{V_{i+1j-1}}}(\tau+\delta, \overline{A}_{V_{i+1j-1}})$, where $\overline{A}_{H_{ij}}$, $\overline{A}_{V_{i+1j-1}}$, $\overline{O}_{H_{ij-1}}$, $\overline{O}_{V_{ij-1}}$ are determined by RS at node $(i, j-1)$.*

Proposition 7. *A needle variation $\eta_{A_{V_{ij}}}(\tau, \overline{A}_{V_{ij}})$ determines a jump of $v_{V_{ij}}^{\Delta N}$ and, if $\Delta N_{V_{ij}} = 0$, the following needle variations: $\eta_{O_{V_{i-1j}}}(\tau+\delta, \overline{O}_{V_{i-1j}})$, $\eta_{O_{H_{i-1j}}}(\tau+\delta, \overline{O}_{H_{i-1j}})$, $\eta_{A_{H_{i-1j+1}}}(\tau, \overline{A}_{H_{i-1j+1}})$, where $\overline{A}_{V_{ij}}$, $\overline{A}_{H_{i-1j+1}}$, $\overline{O}_{V_{i-1j}}$, $\overline{O}_{H_{i-1j}}$ are determined by RS at node $(i-1, j)$.*

Proposition 8. *A needle variation $\eta_{A_{H_{ij}}}(\tau, \overline{A}_{H_{ij}})$ determines a jump of $v_{H_{ij}}^{\Delta N}$ and, if $\Delta N_{H_{ij}} = 0$, the following needle variations: $\eta_{O_{V_{ij-1}}}(\tau+\delta, \overline{O}_{V_{ij-1}})$, $\eta_{O_{H_{ij-1}}}(\tau+\delta, \overline{O}_{H_{ij-1}})$, $\eta_{A_{V_{i+1j-1}}}(\tau, \overline{A}_{V_{i+1j-1}})$, where $\overline{A}_{H_{ij}}$, $\overline{A}_{V_{i+1j-1}}$, $\overline{O}_{H_{ij-1}}$, $\overline{O}_{V_{ij-1}}$ are determined by RS at node $(i, j-1)$.*

In order to better illustrate the effect of a needle variation, we depict in Fig. 7 the effects on $O_{V_{ij}}$ and those on the dynamics at nodes (i,j) and $(i-1,j)$. Obviously the other nearby nodes are affected in a similar manner. To complete the discussion of the cascade effects of a needle variation, we state the following results, omitting the easy proofs for sake of space.

Proposition 9. *If, for some time $\bar{t}$, $\Delta N_{V_{ij}}(t) > 0$ in a left neighborhood of $\bar{t}$, $\Delta N_{V_{i}j}(t) = 0$ in a right neighborhood of $\bar{t}$, and $v_{V_{ij}}^{\Delta N}(\bar{t}) \neq 0$, then it occurs the switching variation $\eta_{\varepsilon_{V_{ij}}}(\bar{t}, 0, -1)$. Similarly switching variations occur in the following cases:*

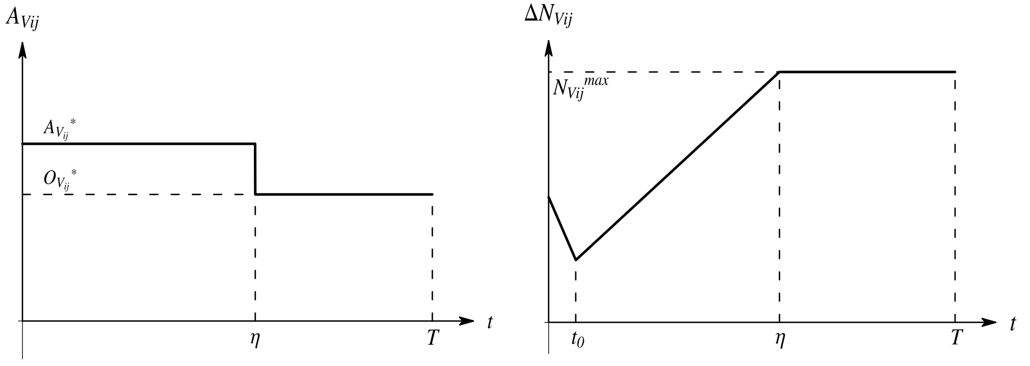

Fig. 8. Evolution of $A_{V_{ij}}$ and $\Delta N_{V_{ij}}(t)$ when all permeability parameters are constant.

- $\Delta N_{Vij}(t) = 0$ *in a left neighborhood of* $\bar{t}$, $\Delta N_{V_{ij}}(t) > 0$ *in a right neighborhood of* $\bar{t}$ *and* $v^{\Delta N}_{V_{ij}}(\bar{t}) \neq 0$;
- $\Delta N_{Vij}(\bar{t}) < \Delta N^{\max}_{Vij}$ *in a left neighborhood of* $\bar{t}$, $\Delta N_{V_{ij}}(t) = \Delta N^{\max}_{Vij}$ *in a right neighborhood of* $\bar{t}$ *and* $v^{\Delta N}_{V_{ij}}(\bar{t}) \neq 0$;
- $\Delta N_{Vij}(t) = \Delta N^{\max}_{Vij}$ *in a left neighborhood of* $\bar{t}$, $\Delta N_{Vij}(\bar{t}) < \Delta N^{\max}_{Vij}$ *in a right neighborhood of* $\bar{t}$ *and* $v^{\Delta N}_{V_{ij}}(\bar{t}) \neq 0$.

Proposition 10. *A switching variation* $\eta_{\varepsilon_{V_{ij}}}(\tau, \varepsilon_1, \varepsilon_2)$ *determines needle variations* $\eta_{A_{V_{ij}}}(\tau, \overline{A}_{V_{ij}})$ *and* $\eta_{O_{V_{ij}}}(\tau, \overline{O}_{V_{ij}})$*, where* $\overline{A}_{V_{ij}}$ *and* $\overline{O}_{V_{ij}}$ *are determined by RS at node* (i, j)*.*

Proposition 11. *A needle variation* $\eta_{\varepsilon_{H_{ij}}}(\tau, \varepsilon_1, \varepsilon_2)$ *determines needle variations* $\eta_{A_{H_{ij}}}(\tau, \overline{A}_{H_{ij}})$ *and* $\eta_{O_{H_{ij}}}(\tau, \overline{O}_{H_{ij}})$*, where* $\overline{A}_{H_{ij}}$ *and* $\overline{O}_{H_{ij}}$ *are determined by RS at node* (i, j)*.*

6. Numerical results

To test numerically the results of the previous section, we run some simulations for a Barcelona network, focusing on two nodes $(i-1, j)$ and (i, j) as in Fig. 7. A fourth order Runge–Kutta scheme is used, with temporal step $h = 0.01$ and a total simulation time $T = 30$. We assume that: $L_{V_{i-1j}} = 6$, $L_{H_{i-1j}} = 5$, $L_{H_{i-1j+1}} = L_{V_{ij}} = L_{H_{ij+1}} = 4$, $L_{H_{ij}} = L_{V_{i+1j}} = 3$; $\alpha_{ij} = \alpha_{i-1j} = \beta_{ij} = \beta_{i-1j} = 0.3$; for all roads, $V_0 = c = 2$, $\rho_{\max} = 1$, hence $\widehat{Q} = 1$; incoming fluxes:

$$A_{V_{i-1j}}(t) = A_{H_{i-1j}}(t) = A_{H_{ij}}(t) = \begin{cases} 0.5 < \widehat{Q} & t \geqslant 0, \\ 0 & \text{otherwise}; \end{cases}$$

initial condition for queues on roads: $\Delta N_{V_{i-1j}}(0) = 3$, $\Delta N_{H_{i-1j}}(0) = \Delta N_{H_{i-1j+1}}(0) = \Delta N_{V_{ij}}(0) = \Delta N_{H_{ij+1}}(0) = 2$, $\Delta N_{H_{ij}}(0) = \Delta N_{V_{i+1j}}(0) = 1$. We distinguish two types of simulations according to permeability parameters.

Assume first that all permeability parameters are constant, namely $\gamma_{V_{i-1j}} = \gamma_{H_{i-1j}} = \gamma_{V_{ij}} = \gamma_{H_{ij}} = 0.5$, $\gamma_{H_{i-1j+1}} = \gamma_{H_{ij+1}} = 0.3$, $\gamma_{V_{i+1j}} = 0.7$. We depict in Fig. 8 the evolution of $A_{V_{ij}}(t)$ and $\Delta N_{V_{ij}}(t)$. For $t > t_0 = \frac{L_{V_{ij}}}{V_0}$, $\Delta N_{V_{ij}}(t)$ increases and reaches the maximal value, $N^{\max}_{V_{ij}}$, at $\eta \simeq 17$. After η, $A_{V_{ij}}(t)$ is equal to $O_{V_{ij}}(t - \frac{L_{V_{ij}}}{c})$, which is constant and equal to $O^*_{V_{ij}} = \gamma_{V_{ij}} \widehat{Q} = 0.5$. Notice that $\Delta N_{V_{ij}}(t)$ is always at the maximal value for $t \geqslant \eta$. The perturbation of $A_{V_{ij}}(t)$ does not affect $\Delta N_{V_{ij}}(t)$ as $A_{V_{ij}}(t - t_0) - O_{V_{ij}}(t) = 0$ for $t \geqslant \eta + t_0$.

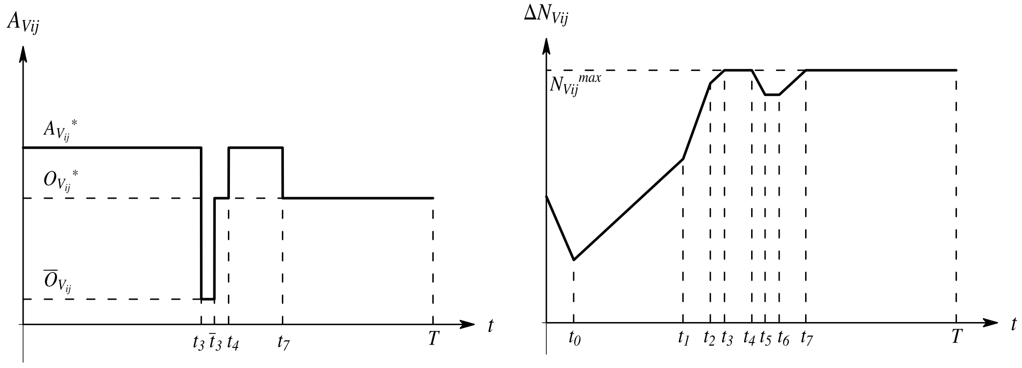

Fig. 9. Evolution of $A_{V_{ij}}$ and $\Delta N_{V_{ij}}(t)$ when a needle variation occurs for $\gamma_{V_{ij}}$.

Assume now constant permeability parameters with the exception of $\gamma_{V_{ij}}(t)$, for which a needle variation occurs, namely: $\gamma_{V_{i-1j}} = \gamma_{H_{i-1j}} = \gamma_{H_{ij}} = 0.5$, $\gamma_{H_{i-1j+1}} = \gamma_{H_{ij+1}} = 0.3$, $\gamma_{V_{i+1j}} = 0.7$,

$$\gamma_{V_{ij}}(t) = \begin{cases} \gamma^{*}_{V_{ij}} & t \in [0, t_1] \cup]t_2, T], \\ \omega_{V_{ij}} & t \in]t_1, t_2], \end{cases}$$

where $\gamma^{*}_{V_{ij}} = 0.5$, $\omega_{V_{ij}} = 0.1$, $t_1 = 10$ and $t_2 = 12$. We depict in Fig. 9 the evolutions of $A_{V_{ij}}(t)$ and $\Delta N_{V_{ij}}(t)$. For $t > t_0$, $\Delta N_{V_{ij}}(t)$ increases, as $A_{V_{ij}}(t - t_0) - O_{V_{ij}}(t) > 0$. At t_1, the needle variation for $\gamma_{V_{ij}}(t)$ generates a needle variation for $O_{V_{ij}}(t)$, that causes an immediate change of slope for $\Delta N_{V_{ij}}(t)$. At t_2, the needle variation for $\gamma_{V_{ij}}(t)$ vanishes, hence $\Delta N_{V_{ij}}(t)$ changes its own slope again. At $t_3 \simeq 13$, $\Delta N_{V_{ij}}(t)$ reaches its maximal value and $A_{V_{ij}}(t)$ follows $O_{V_{ij}}(t - \frac{L_{V_{ij}}}{c})$. Notice that, due to the needle variation of $\gamma_{V_{ij}}(t)$,

$$O_{V_{ij}}\left(t - \frac{L_{V_{ij}}}{c}\right) = \begin{cases} \overline{O}_{V_{ij}} & \text{if } t_1 < t - \frac{L_{V_{ij}}}{c} \leqslant t_2, \\ O^{*}_{V_{ij}} & \text{otherwise}, \end{cases}$$

where $\overline{O}_{V_{ij}} = \omega_{V_{ij}} \widehat{Q} = 0.1$ and $O^{*}_{V_{ij}} = \gamma^{*}_{V_{ij}} \widehat{Q} = 0.5$. Hence $A_{V_{ij}}(t)$ is equal to $\overline{O}_{V_{ij}}$ for $t \in [t_3, \bar{t}_3]$, $\bar{t}_3 = t_2 + t_0$, and equal to $O^{*}_{V_{ij}}$ for $t \in]\bar{t}_3, t_4]$, $t_4 = t_3 + t_0$. At t_4, $\Delta N_{V_{ij}}(t)$ starts to decrease as $A_{V_{ij}}(t_4 - t_0) - O_{V_{ij}}(t_4) < 0$, and $A_{V_{ij}}(t)$ assumes the value imposed by the solution of dynamics at node $(i-1, j)$, $A^{*}_{V_{ij}}$; $\Delta N_{V_{ij}}(t)$ becomes constant for $t \in [t_5, t_6[$, $t_5 = \bar{t}_3 + t_0$, $t_6 = t_4 + t_0$, as $A_{V_{ij}}(t - t_0) - O_{V_{ij}}(t) = 0$. At t_6, $\Delta N_{V_{ij}}(t)$ starts to increase, and it grows until $t_7 \simeq 19$, where the maximal value is achieved. As $\Delta N_{V_{ij}}(t) = N^{\max}_{V_{ij}}$, $A_{V_{ij}}(t) = O_{V_{ij}}(t - \frac{L_{V_{ij}}}{c})$. Notice that $\Delta N_{V_{ij}}(t)$ remains equal to $N^{\max}_{V_{ij}}$, as $A_{V_{ij}}(t - t_0) - O_{V_{ij}}(t) = 0$ for $t \geqslant t_8 = t_7 + t_0$.

7. Conclusions

We analyzed a delayed-ode approach to describe vehicular traffic flows on networks of Barcelona type. The model is rigorously derived from the classical LWR ones, via recent results for networks.

Minimization of queue lengths is considered, in terms of permeability parameters which regulate the inflow at nodes. Unfortunately, the overall dynamics gives rise to nested equations with variable delays. Therefore, logic variables were introduced and a hybrid framework was thus obtained.

Sensitivity analysis for permeability parameters was developed, based on needle variations. The multiple effects of such variations were first described by theoretical results and then verified by simulations.

Further research should be developed to get more information on optimal controls, e.g. using necessary conditions for hybrid control systems. Unfortunately available results do not yet cover delayed hybrid equations.

From numerical point of view, large scale simulations may provide insight on the dynamics of the whole network. A *slower is faster* effect may show up as a consequence of the relationships between inflows and outflows at different nodes.

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